

# A crucial hypothesis for Inflation

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## Abstract

As is well known there are many different inflationary models that can explain the accelerated expansion occurred in the early Universe. It is possible that there exists a fundamental property of this period that could provide the corresponding field theory. Our hypothesis is that the Ricci scalar should assume a positive constant value during Inflation. Considering a single scalar field that drives the inflationary phase, we obtain uniquely the scalar potential and the coupling term between the scalar field and spacetime geometry. For this potential, that is well known in the inflationary scenario, the theoretical prediction of the spectral index  $n_s$  is in perfect agreement with the experimental data. Interestingly in two dimensions, our model is equivalent to Liouville gravity.

# 1 Introduction

The aim of Cosmology is to explain the origin, dynamics and evolution of the Universe at large scales. For large scales we mean those for which galaxies and clusters of galaxies can be considered as point-like objects. A cosmological model is a model that attempts to explain the dynamics of these material points in agreement with astronomical observations. The most powerful model is the Cosmological Standard Model. Its basic assumption is that at large scales and at all times of expansion of the Universe, the matter (regarded as a perfect fluid) is spatially distributed in a homogeneous and isotropic manner.

At these scales, gravity is the dominant force and it is described by Einstein's General Relativity and the space-time metric is the Friedmann-Robertson-Walker (FRW) metric. The Cosmological Standard Model explains the abundance of light elements, the recession of galaxies, the thermodynamic state of the observable universe. However, it also has some inconsistencies as the problem of homogeneity, flatness, and that concerns the birth and formation of large scale structures.

A. Guth realized [1], that considering in the early Universe a very short period during which the scale factor expands exponentially, it was possible to solve the three problems mentioned. This phase is called the inflationary phase. To obtain this expansion, we must go beyond the assumption of perfect fluid for matter. It is known that at high temperatures the classical description of matter as an ideal gas is not valid. We expect that a fair description of the early Universe can be determined using a valid theory of matter at high energies. One possibility is to use a scalar field. In this context an inflation model must explain the anisotropies of the CMB (the Cosmic Microwave Background radiation) and the reheating [2]. The reheating is the process by which inflation ends. Of course the end of inflation is crucial because only in a period of the Universe when the expansion is not accelerated we can have the baryogenesis.

Thus over the years a myriad of inflationary theories have been developed [3], with many different self-interaction potentials and there is not yet a standard inflationary model.

The goal in our work is to start with a precise assumption on Inflation, obtaining uniquely the relative scalar field action. We consider a single scalar field that drives Inflation and the crucial hypothesis is that in the inflationary

period, the spacetime lorentzian manifold has a positive constant Ricci scalar. The demand for a positive constant Ricci scalar is not a request so unusual. The inflationary phase is in fact also called de Sitter phase. In a pure de Sitter phase, a cosmological constant dominates the expansion dynamics and the spacetime is called de Sitter spacetime (vacuum solution of Einstein's field equations with a positive cosmological constant). In this spacetime the Ricci scalar is constant. We generalize this propriety about Ricci scalar to a spacetime where there is also a scalar field.

In the next section, we derive uniquely the coupling term between the scalar field and the Ricci scalar and the inflationary potential. This scalar potential is the double-well potential and it is well known in the study of inflation [7] [8] [9] [10] [11] [12] [13]. In this model, the theoretical prediction of the spectral index is in perfect agreement with its experimental value [4] [10] [11] [12].

Finally we show that in the limit  $D = 2$  the model is equivalent to Liouville gravity, an important model of two dimensional gravity that derives from the conformal anomaly of non-critical string theory [5].

## 2 The Model

Let's consider the action of a single scalar field in D dimensions:

$$S_D = \int d^D x \sqrt{-g} \left[ \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + f(\phi) R + U(\phi) \right] \quad (1)$$

where  $g_{\mu\nu}$  is a pseudo-riemannian metric,  $f(\phi)R$  is the coupling term between the scalar field and spacetime and  $U(\phi)$  is the scalar potential.

With this action, we derive the equations of motion:

$$\square \phi - f'(\phi) R - U'(\phi) = 0 \quad (2)$$

$$\begin{aligned} f(\phi) \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} g_{\mu\nu} \partial^\gamma \phi \partial_\gamma \phi - f'(\phi) \partial_\mu \partial_\nu \phi \\ - f''(\phi) \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} f'(\phi) \square \phi + g_{\mu\nu} f''(\phi) \partial^\gamma \phi \partial_\gamma \phi - \frac{1}{2} g_{\mu\nu} U(\phi) = 0 \end{aligned} \quad (3)$$

where  $f'(\phi) = \frac{d}{d\phi}f(\phi)$  and  $U'(\phi) = \frac{d}{d\phi}U(\phi)$ .

The first equation is the generalized Klein-Gordon equation. We calculate the trace of the second equation and after some manipulations, we obtain

$$\square\phi - \frac{\left(\frac{D-2}{2}\right) f(\phi)R + \left(\frac{D}{2}\right) U(\phi)}{(D-1)f'(\phi)} + \frac{\left[(D-1)f''(\phi) - \frac{D-2}{4}\right] \partial^\gamma\phi\partial_\gamma\phi}{(D-1)f'(\phi)} = 0 \quad (4)$$

Our model is based on the hypothesis that during the inflationary period, the Ricci scalar  $R$  assumes a positive constant value:

$$R = \alpha^2 \quad (5)$$

then we consider only a subclass of metrics of  $g_{\mu\nu}$ , solution of equations (2) and (3), that admits this condition.

Supposing to know explicitly the metric and replacing  $R$  with  $\alpha^2$  in (2) and (4) thanks to the relation (5), we have a system of two partial differential equations for the same field  $\phi$ . A simple way, for being sure that the system admit a solution, is to consider the equations (2) and (4) as the same equation. We have the following conditions:

$$(D-1)f''(\phi) - \frac{D-2}{4} = 0 \quad (6)$$

$$f'(\phi)\alpha^2 + U'(\phi) = \frac{\left(\frac{D-2}{2}\right) f(\phi)\alpha^2 + \left(\frac{D}{2}\right) U(\phi)}{(D-1)f'(\phi)} \quad (7)$$

and the relative solutions are respectively:

$$f(\phi) = \frac{1}{4} \left( \frac{D-2}{D-1} \right) \frac{\phi^2}{2} + \beta\phi + \delta \quad (8)$$

$$U(\phi) = C \left[ \left( \frac{D-2}{2D} \right) \phi + 2 \left( \frac{D-1}{D} \right) \beta \right]^{\frac{2D}{D-2}} + \alpha^2 \left[ 2 \left( \frac{D-1}{D} \right) \beta^2 - \left( \frac{D-2}{D} \right) \delta \right] \quad (9)$$

where  $\beta$ ,  $\delta$  and  $C$  are generic constants and  $\xi = \frac{1}{4} \left( \frac{D-2}{D-1} \right)$  is the conformal coupling in  $D$  dimensions.

Placing  $\beta = \frac{(4-D)(3-D)}{2} \beta'$  (where  $\beta'$  is another constant), we get the following identifications:

$$\delta = -\frac{1}{16\pi G} \quad \alpha^2 = \frac{2D}{D-2} \Lambda \quad (10)$$

where  $G$  is the Newton's constant and  $\Lambda$  is a cosmological constant ( $\Lambda$  has not the same numerical value of the cosmological constant of  $\Lambda$ CDM model). The action (1) for  $D = 4$  is

$$S_4 = \int d^4x \sqrt{-g} \left[ -\frac{1}{16\pi G} (R - 2\Lambda) + \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{1}{12} \phi^2 R - C' \phi^4 \right] \quad (11)$$

where  $C' = -\frac{C}{4^4}$

Thus we have determined exactly the total action for our scalar field.

Moreover the coupling term  $\phi^2 R$  and the Einstein-Hilbert term represent two potential terms because the Ricci scalar is constant and we can replace it with  $\alpha^2$ . Thus in the action  $S_4$  there is a kinetic term and a total potential  $V(\phi)$ .

The Lagrangian density is  $\mathcal{L} = \sqrt{-g} \left[ \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right]$  with

$$V(\phi) = \frac{\Lambda}{8\pi G} - \frac{\Lambda}{3} \phi^2 + C' \phi^4 \quad (12)$$

where we have used the relation (10) between  $\alpha^2$  and  $\Lambda$ .

In the context of Cosmological Standard Model, we consider the FRW flat metric:  $ds^2 = -dt^2 + a(t)^2 d\vec{x}^2$ , where  $a(t)$  is the scale factor.

With this metric, the equations (5) and (2) become

$$R = 6 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right] = 6(2H^2 + \dot{H}) = \alpha^2 \quad \ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0 \quad (13)$$

where  $H = \frac{\dot{a}}{a}$

A solution of the first relation gives us the scale factor  $a(t)$  that represents the accelerated expansion:

$$a(t) = a_0 \sqrt{\cosh \sqrt{\frac{4\Lambda}{3}} t} \quad (14)$$

where  $a_0 = a(0)$ .

Moreover if we introduce the (non standard) Planck energy scale  $M_P = \sqrt{\frac{3}{4\pi G}}$  ( $c = \hbar = 1$ ) and the energy scale of Inflation  $M_I = \alpha = \sqrt{4\Lambda}$  and considering  $C' = \frac{1}{4!} \left(\frac{M_I}{M_P}\right)^2$ , we can rewrite the potential (12) as

$$V(\phi) = \frac{1}{4!} (M_I M_P)^2 \left[ 1 - \left( \frac{\phi}{M_P} \right)^2 \right]^2 \quad (15)$$

and the hierarchy condition  $M_I \ll M_P$  [6] guarantees that  $C' \ll 1$ .

This scalar potential is well known in the inflationary scenario. It is the double-well (Landau-Ginzburg) potential and it has been studied in different approaches [7] [8] [9] [10] [11] [12] [13].

In this model, the fundamental result in the context of slow-roll Inflation [2], is that the theoretical prediction of the spectral index  $n_s$  [10] [11] [12] is in perfect agreement with its experimental value [4].

Our final consideration concerns the fact that in (9) for  $D = 2$  and fixing  $\beta = 1$ , there is the finite limit

$$\lim_{D \rightarrow 2} \left[ \left( \frac{D-2}{2D} \right) \phi + \frac{(D-1)(4-D)(3-D)}{D} \right]^{\frac{2D}{D-2}} = e^\phi \quad (16)$$

and then in two dimensions, the total action is

$$S_2 = \int d^2x \sqrt{-g} \left[ \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \phi R + C e^\phi + \alpha^2 \right] \quad (17)$$

It coincides with the action of Liouville gravity that is an important action in the context of non-critical string theory and it is also a relevant model of

gravity in two dimensions [5]. This link with the Liouville gravity could be an first indication of a deeper connection between our phenomenological theory and a more fundamental theory.

In conclusion, starting by a simple hypothesis about spacetime geometry during the inflationary period, we have obtained the precise action of scalar field with a specific potential, where the theoretical prediction of the spectral index  $n_s$  is in perfect agreement with the experimental data.

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